# ON GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR THE VLASOV-POISSON SYSTEM IN CONVEX BOUNDED DOMAINS

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ABSTRACT. We prove global existence of strong solutions for the Vlasov-Poisson system in a convex bounded domain in the plasma physics case assuming homogeneous Dirichlet boundary conditions for the electric potential and the specular reflection boundary conditions for the distribution density.

## 1. INTRODUCTION

We consider the Vlasov-Poisson system in a smooth, convex and bounded domain $\Omega$, with the reflection boundary condition given by

$$(1) \qquad \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \quad \text{for } (t, x, v) \in [0, \infty) \times \Omega \times \mathbb{R}^3,$$

$$(2) \qquad \Delta \phi(t, x) = \rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\, dv, \quad \text{for } (t, x) \in [0, \infty) \times \Omega,$$

$$(3) \qquad f(0, x, v) = f_0(x, v), \text{ for } (x, v) \in \Omega \times \mathbb{R}^3$$

$$(4) \qquad f(t, x, v) = f(t, x, v^*), \quad \text{for } x \in \partial\Omega,$$

where $v^* = v - 2(v \cdot n_x)n_x$, $n_x$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$, and we assume $\Omega$ has a $C^5$ boundary. Here $f(t, x, v) \geq 0$ represents the distribution density of electrons and $\phi$ is the electrostatic potential, $f_0(x, v)$ is the prescribed initial datum, and $\rho(t, x)$ is the macroscopic charge density. We impose the Dirichlet boundary condition for the electric potential $\phi$,

$$(5) \qquad \phi(t, x) = 0 \quad \text{if } x \in \partial\Omega.$$

In this paper, we aim to understand the role of boundaries in the dynamics of kinetic models, in particular the Vlasov-Poisson system and to develop tools for boundary-value problems in kinetic models.

For the whole space without a boundary in one and two dimensions, a smooth solution is known to exist globally in time [13], [19]. For the three dimensional case, Batt [2], Horst [8], and Bardos-Degond [1] proved the global existence of classical solutions for spherical, cylindrically symmetric,

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and general but small initial data respectively. In the case of $\Omega = \mathbb{R}^3$ with arbitrary initial data, the solutions of the system (1)-(4) are globally defined in time, as it was proved in [16] as well as in [15] using different methods.

However, in the presence of boundaries, the mathematical theory of well-posedness for the solutions of the Vlasov-Poisson system becomes more complicated compared to the case of the whole space. It was proved in [6] that classical solutions for the Vlasov-Poisson system may not exist in general without the nonnegativity assumption if $\Omega$ is the half-space $\mathbb{R}^3_+$. On the other hand, it was also proved in [6] that even with the nonnegativity assumption the derivatives of the solutions of (1)-(4) cannot be uniformly bounded near the boundary of $\Omega$ due to the fact that a Lipschitz estimate for the characteristics in terms of the initial data is not possible.

One of the main difficulties in order to solve (1)-(5), even for short times, is to keep track of the evolution of the characteristic curves associated to (1) which remain close during their evolution to the so-called singular set, defined as follows

(6) $$\Gamma = \left\{ (x, v) \in \Omega \times \mathbb{R}^3 : x \in \partial\Omega, \ v \in T_x \partial\Omega \right\},$$

where $T_x \partial\Omega \subset \mathbb{R}^3$ is the tangent plane to $\partial\Omega$ at the point $x$.

Boundary-value problems should be treated more carefully and difficulties due to singularity formation at a boundary may be expected [6]. Global existence in a half-space of solutions of (1)-(3) satisfying the specular reflection boundary condition (4) and Neumann boundary condition was first proved by Guo (cf. [7]) by adapting a high velocity moment method in [15]. The proof of the global existence of solutions of (1)-(3) satisfying (4) and the Dirichlet boundary condition (5) was recently proved in [11]. For general convex bounded domains but with the Neumann boundary condition for $\phi$, the global well-posedness was recently shown in [12].

However, a global existence theory of the Dirichlet boundary problem for the electric potential $\phi$ has not been given yet and this paper is devoted to proving the global existence of solutions to the Vlasov-Poisson system (1)-(3) with the specular reflection boundary condition (4) for $f$ and the Dirichlet boundary condition (5) for $\phi$.

This paper combines the methods in the papers [11] and [12] to prove global existence of solutions for the Vlasov-Poisson system in arbitrary smooth convex domains with Dirichlet boundary conditions. The analysis in [12] allows to study problems with Neumann boundary conditions. This is due to the fact that an essential ingredient of the argument in [12] is the velocity Lemma first proved in [6] which shows that the characteristic curves associated to (1)-(2) cannot approach to the so-called singular set if initially they are outside of it. The argument allows to generalize such type of velocity Lemmas to Dirichlet boundary conditions, which was obtained in [11], but for the half-space case. However, simple adaptations of such techniques in the half-space problem to the case of arbitrary convex bounded

domains do not work. The difficulty is that, contrary to the half-space case [11], we cannot have the representation formula for $\phi$, due to the incapability of finding an explicit form of the Green function for a general convex domain $\Omega$. As we will see, this problem could be settled mainly by applying refined boundary estimates for the Laplace operator, and constructing relevant supersolutions.

This paper proves global existence of solutions for the Vlasov-Poisson system for arbitrary smooth domains with Dirichlet boundary conditions. The main new contents of this paper are some technical estimates that allow to extend the arguments of [11] to arbitrary smooth convex domains. These estimates require detailed control of the newtonian potential with Dirichlet boundary conditions as well as some of its derivatives at points close to the boundary of the domain. The combination of these methods with ones in [12] allows to prove the stated global existence results.

The paper is organized as follows. Preliminary notations and main result of the global existence will be presented in Section 2 and Section 3 is devoted to the Velocity lemma and the corresponding linear problem. In Section 4, an iterative scheme for the nonlinear problem is investigated and finally the global bound on a key quantity for the global existence is obtained in Section 5.

## 2. Main Result

First we fix a point $x \in \Omega$ and denote it as $\widetilde{x}$ to indicate our target point. Notice that our goal is to see whether a trajectory starting from the singular set $\{(x, v) \in \partial\Omega \times \mathbb{R}^3; v \cdot n_x = 0\}$ propagates into the interior $\Omega \times \mathbb{R}^3$. So, we may assume that $\widetilde{x}$ is near the boundary $\partial\Omega$. Let $x_0$ be the boundary point closest to $\widetilde{x}$. By proper rotations and translations, we may set $x_0 = (0, 0, 0)$, $\widetilde{x} = (\widetilde{x}_1, 0, 0)$, and $\Omega \subset \mathbb{R}^3_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 > 0\}$ so that the tangent plane to $\partial\Omega$ at $x_0$ is just $\{x_1 = 0\}$.

Now, we use the local parametrization near $x_0$ to define

$$x = x_\parallel(\mu_1, \mu_2) - x_\perp n(\mu_1, \mu_2),$$

so that $x_\parallel(\mu_1, \mu_2)$ is the point of $\partial\Omega$ closest to $x$ and $n(\mu_1, \mu_2)$ is the outward normal to $\partial\Omega$ at $x_\parallel$. For this $x$, we represent $v$ by

$$v = v_\parallel(\mu_1, \mu_2) - v_\perp n(\mu_1, \mu_2),$$

where $v_\parallel(\mu_1, \mu_2) = w_1 u_1 + w_2 u_2 \in T_{x_\parallel(\mu_1, \mu_2)}\partial\Omega$ is the tangential component of $v$ and $\{u_1, u_2\}$ is the basis of $T_{x_\parallel(\mu_1, \mu_2)}\partial\Omega$ given by $u_i := \frac{\partial x_\parallel(\mu_1, \mu_2)}{\partial\mu_i}$ for $i = 1, 2$.

The system of coordinates $(\mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp)$ provides a more convenient representation for the set of points in the phase space $\Omega \times \mathbb{R}^3$ that are close to the singular set $\Gamma$ defined in (6) as in [12]. The original equation (1) takes in this new set of coordinates the following form in Lemma below, which is in [12] and we skip its proof.

**Lemma 2.1.** *The equation (1) can be rewritten for $(x, v) \in [\partial\Omega + B_\delta(0)] \times \mathbb{R}^3$, and using the set of coordinates $(\mu_1, \mu_2, x_\perp, w_1, w_2, v_\perp)$ in the form*

$$(7) \qquad \frac{\partial f}{\partial t} + \sum_{i=1}^{2} \frac{w_i}{1 + k_i x_\perp} \frac{\partial f}{\partial \mu_i} + v_\perp \frac{\partial f}{\partial x_\perp} + \sum_{i=1}^{2} \sigma_i \frac{\partial f}{\partial w_i} + F \frac{\partial f}{\partial v_\perp} = 0,$$

*where*

$$(8) \quad \sigma_i \equiv E_i - \frac{v_\perp w_i k_i}{1 + k_i x_\perp} - \sum_{j,\ell=1}^{2} \frac{\Gamma_{j,\ell}^{i} w_j w_\ell}{1 + k_j x_\perp}, \quad F \equiv E_\perp + \sum_{j=1}^{2} \frac{w_j^2 b_j}{1 + k_j x_\perp},$$

*where $k_j$ are the principal curvatures, $b_j$ are the coefficients $e$ and $g$ from the second fundamental form according to the notation in [18] and $\Gamma_{j,\ell}^{i}$ are the Christoffel symbols of the surface $\partial\Omega$. The vector $E = \nabla_x \phi$ has been written in the form*

$$(9) \qquad\qquad E = E_1 u_1 + E_2 u_2 - E_\perp n(\mu_1, \mu_2),$$

*where $u_1, u_2$ are defined above.*

*Remark.* Notice that since the domain $\Omega$ is convex, and due to the nonnegativity assumption we have $F < 0$.

To prove global existence, we need to make some necessary assumptions.
1. **Compatibility conditions for the initial data.**
In order to obtain classical solutions of (1)-(5) we need to impose the following compatibility conditions on the initial data $f_0(x, v)$ at the reflection points of $\partial\Omega \times \mathbb{R}^3$ (cf. [6], [10]).

$$(10) \qquad\qquad\qquad\qquad\qquad f_0(x, v) = f_0(x, v^*),$$

$$(11) \quad v^\perp \left[ \nabla_x^\perp f_0(x, v^*) + \nabla_x^\perp f_0(x, v) \right] + 2E^\perp(0, x) \nabla_v^\perp f_0(x, v) = 0,$$

where $E^\perp(0, x)$ is the decomposition of the field $E(0, x)$ given by (9) and $\nabla_x^\perp$, $\nabla_v^\perp$ are the normal components to $\partial\Omega$ of the gradients $\nabla_x$, $\nabla_v$ respectively.

2. **Flatness condition.**
We assume that $f_0$ is constant near the singular set (cf. [7] as well as [10]). More precisely we will assume that $f_0 \in C^{1,\mu}$ satisfies the following flatness condition near the singular set $\Gamma$

$$(12) \qquad\qquad f_0(x, v) = \text{constant} , \ \text{dist}((x, v), \Gamma) \leq \delta_0$$

for some $\delta > 0$ small.

We need to introduce some functional spaces for technical reasons. We define for $\mu \in (0,1)$,

$$\|f\|_{C^{1,\mu}\left(\bar{\Omega}\times\mathbb{R}^3\right)} = \sup_{(x,v),(x',v')\in\bar{\Omega}\times\mathbb{R}^3} \left(\frac{|\nabla f\left(x,v\right)-\nabla f\left(x',v'\right)|}{|x-x'|^\mu + |v-v'|^\mu}\right) + \|f\|_{L^\infty\left(\bar{\Omega}\times\mathbb{R}^3\right)} \quad, \quad \nabla = (\nabla_x, \nabla_v) ,$$

$$C_0^{1,\mu}\left(\bar{\Omega}\times\mathbb{R}^3\right) = \left\{ f \in C^{1,\mu}\left(\bar{\Omega}\times\mathbb{R}^3\right) : f \text{ compactly supported, } \|f\|_{C^{1,\mu}\left(\bar{\Omega}\times\mathbb{R}^3\right)} < \infty \right\} ,$$

$$\|f\|_{C_{t;\,x}^{1;1,\mu}\left([0,T]\times\bar{\Omega}\right)} \equiv \sup_{x,x'\in\bar{\Omega},\,t,t'\in[0,T]} \frac{|\nabla_x f\left(t,x\right)-\nabla_x f\left(t',x'\right)|}{|x-x'|^\mu}$$
$$+ \|f\|_{C\left([0,T]\times\bar{\Omega}\right)} + \|f_t\|_{C\left([0,T]\times\bar{\Omega}\right)} \quad,$$

$$\|f\|_{C_{t;(x,v)}^{1;1,\mu}([0,T]\times\Omega\times\mathbb{R}^3)}$$
$$\equiv \sup_{x,x'\in\bar{\Omega},\,t,t'\in[0,T]} \frac{|\nabla_x f\left(t,x,v\right)-\nabla_x f\left(t',x',v\right)| + |\nabla_v f\left(t,x,v\right)-\nabla_v f\left(t',x',v'\right)|}{|x-x'|^\mu + |v-v'|^\mu}+$$
$$+ \|f\|_{C\left([0,T]\times\bar{\Omega}\times\mathbb{R}^3\right)} + \|f_t\|_{C\left([0,T]\times\bar{\Omega}\times\mathbb{R}^3\right)} \cdot$$

We define the spaces $C\left([0,T]\times\bar{\Omega}\right)$, $C\left([0,T]\times\bar{\Omega}\times\mathbb{R}^3\right)$ as the spaces of continuous functions bounded in the uniform norm.

### The main result: Global existence Theorem.

The main result of this paper is the following.

**Theorem 2.2.** *Let* $f_0 \in C_0^{1,\mu}\left(\Omega\times\mathbb{R}^3\right)$ *for some* $0 < \mu < 1$ *with* $f_0 \geq 0$ *and let* $f_0$ *satisfy (10)-(12). Then there exists a unique solution* $f \in C_{t;(x,v)}^{1;1,\lambda}\left((0,\infty)\times\Omega\times\mathbb{R}^3\right)$, $\phi \in C_{t;x}^{1;3,\lambda}\left([0,\infty)\times\Omega\right)$, *for some* $0 < \lambda < \mu$, *of the Vlasov-Poisson system (1)-(5) with compact support in* $x$ *and* $v$.

## 3. Velocity Lemma and Linear Problem

Next, we introduce the evolution of characteristic curves associated to the Vlasov-Poisson system with the specular reflection at the boundary.

Let $E(t,x) := \nabla_x \phi(t,x)$ be given. We define $(X(s;t,x,v), V(s;t,x,v)) \in \overline{\Omega} \times \mathbb{R}^3$ such that for each $(x,v) \in \Omega \times \mathbb{R}^3$,

$$(13) \qquad \frac{dX}{ds}(s;t,x,v) = V(s;t,x,v),$$

$$(14) \qquad \frac{dV}{ds}(s;t,x,v) = E(s, X(s;t,x,v)) = \nabla_x \phi(s, X(s;t,x,v)),$$

$$(15) \qquad (X(t), V(t)) = (x,v),$$

as long as $X \in \Omega$. The reflection boundary condition says that if $X(s_1; t, x, v) \in \partial\Omega$ for some $s_1 \in [0, T]$, then
(16)
$$V(s_1^+; t, x, v) = \lim_{\substack{s \to s_1 \\ s > s_1}} V(s; t, x, v) = (V(s_1^-; t, x, v))^* = \left( \lim_{\substack{s \to s_1 \\ s < s_1}} V(s; t, x, v) \right)^*.$$
Here, $V^* = V - 2(V \cdot n_X)n_X$ where $n_X$ is the outward unit normal vector to $\partial\Omega$ at $X$.

Before giving an explicit formulation, we consider some underlying motivations. If we rephrase the Velocity Lemma, it is equivalent to saying that a trajectory starting near the singular set $\{x_\perp = v_\perp = 0\}$ remains near it in the future. More precisely, by using the local coordinates, we represent the normal component of the characteristic equations from Lemma 2.1 by

$$\frac{dx_\perp}{dt} = v_\perp, \qquad \frac{dv_\perp}{dt} = E_\perp(t, x) + \sum_{i=1}^{2} \frac{w_i^2 b_i}{1 + k_i x_\perp},$$

where $E_\perp(t, x)$ is the normal component of $E(t, x)$, $k_i$'s are the principal curvatures, and $b_i$'s are the coefficients $e$ and $g$ from the second fundamental form, according to the notations in [18]. Notice that a trajectory cannot escape from the singular set, provided $\dot{v}_\perp < 0$ near the boundary. Roughly, this is true because $E_\perp < 0$ due to Hopf Lemma and $b_i \le 0$ by the convexity of $\Omega$. Finally, we define a Lyapunov function

$$\alpha(t, x, v) := \frac{v_\perp^2}{2} - \phi(t, x) - \sum_{i=1}^{2} \frac{w_i^2 b_i}{1 + k_i x_\perp} x_\perp$$

and confirm the stability by differentiating it along the trajectory. It may be possible to choose another function which is equivalent to $x_\perp + v_\perp^2$, but the functional $\alpha$ makes computations simpler because of the cancellations.

Now, we begin to show the following Lemmas which will be the crucial estimations required to derive our main result. Recall that $\Omega$ and $\widetilde{x}$ are given in Section 2.

**Lemma 3.1.** *Let $T > 0$. Suppose that $\phi(t, x)$ solves the following boundary value problem*

$$\Delta\phi(t, x) = \rho(t, x) \quad for \ (t, x) \in [0, T] \times \Omega,$$
$$\phi(t, x) = 0 \quad if \ x \in \Omega,$$

*where $\rho \in C^1([0, T] \times \Omega)$ is given by*
(17)
$$\partial_t \rho + \nabla \cdot j = 0 \quad in \ [0, T] \times \Omega,$$
*for some $j \in (C^1([0, T] \times \Omega))^3$. Then we have*

$$\left| \frac{\partial\phi}{\partial t}(t, \widetilde{x}) \right| \le C\widetilde{x}_1(1 + |\log \widetilde{x}_1|),$$

*where $C > 0$ depends only on $L := \text{diam}\,\Omega$ and $\|j\|_\infty$.*

*Proof.* Let $R = |\widetilde{x}| \ll 1$. We change the variables $x$ and $y$ by $X = \frac{x}{R}$ and $Y = \frac{y}{R}$. Also, $\widetilde{X} = \frac{\widetilde{x}}{R} = (1,0,0)$. Let $G$ be the Green function for the given domain $\Omega$. Then, by the representation formula and (17), it is sufficient to show that

$$(18) \qquad \int_{\frac{\Omega}{R}} |\nabla_Y G(\widetilde{X}, Y)|\, dY \le \frac{C}{R}(1 + |\log R|).$$

We split the region of integration into several parts.

**Case 1.** If $|Y| \le 4$, then we decompose $G(X,Y) = \overline{G}(X,Y) + W(X,Y)$ where

$$\overline{G}(X,Y) = -\frac{1}{4\pi R}\left( \frac{1}{|X - Y|} - \frac{1}{|X^* - Y|} \right)$$

is the Green function for the Half-space restricted to $\frac{\Omega}{R} \times \frac{\Omega}{R}$. Here, $X^*$ represents the reflection of $X$ with respect to the plane $\{X_1 = 0\}$. Clearly, we have

$$\int_{|Y| \le 4} |\nabla_Y \overline{G}(X,Y)|\, dY \le \frac{C}{R}.$$

On the other hand, let $\Psi = \nabla_Y W$. Then, $\Psi$ satisfies

$$\Delta_X \Psi(X,Y) = 0 \quad \text{for } X, Y \in \frac{\Omega}{R},$$

$$\Psi(X,Y) = -\frac{1}{4\pi R}\nabla_Y\left( \frac{1}{|X - Y|} - \frac{1}{|X^* - Y|} \right) \quad \text{for } X \in \frac{\partial\Omega}{R}.$$

Firstly, if $\text{dist}(Y, \frac{\partial\Omega}{R}) \ge 1$, then for all $X \in \frac{\partial\Omega}{R}$, we have

$$|\Psi(X,Y)| \le \frac{C|X - X^*|}{R|X - Y|^3} \le \frac{C|X|^2}{|X|^3 + 1} \le C,$$

by Taylor Theorem and the quadratic approximations of $\frac{\partial\Omega}{R}$. Notice that the constant $C > 0$ can be chosen uniformly with respect to $Y$. Thus,

$$|\Psi(X,Y)| \le C \quad \text{for all } X \in \frac{\Omega}{R},$$

by the maximum principle.

Secondly, if $\text{dist}(Y, \frac{\partial\Omega}{R}) \le 1$, let $Y_0 \in \frac{\partial\Omega}{R}$ be the boundary point closest to $Y$, i.e., $\text{dist}(Y, Y_0) = \text{dist}(Y, \frac{\partial\Omega}{R})$. Then for $X \in \frac{\partial\Omega}{R}$, we get

$$|\Psi(X,Y)| \le \frac{C}{R|X - Y|^2} \quad \text{if } |X - Y_0| \ge 1 \text{ or } \le CR,$$

by the triangle inequality and the convexity of $\Omega$. And, we have

$$|\Psi(X,Y)| \le \frac{C}{|X - Y_0|^3} \quad \text{if } CR \le |X - Y_0| \le 1,$$

by Taylor Theorem. Putting these together, we let

$$
\widetilde{\Psi}(X,Y) = \begin{cases} \dfrac{C}{R} & \text{if } |X - Y_0| \geq 1, \\[2mm] \dfrac{C}{R|X - Y|^2} & \text{if } |X - Y_0| \leq CR, \\[2mm] \dfrac{C}{|X - Y_0|^3} & \text{if } CR \leq |X - Y_0| \leq 1. \end{cases}
$$

Now, for $|Y| \leq 4$ and $\text{dist}(Y, \frac{\partial\Omega}{R}) \leq 1$, by the maximum principle, we construct a supersolution via the Poisson integral formula,

$$
|\Psi(\widetilde{X}, Y)| \leq \int_{\frac{\partial\Omega}{R}} \frac{1}{(1 + |\xi|^2)^{\frac{3}{2}}} \widetilde{\Psi}(\xi, Y)\, d^2\xi
$$

$$
= \int_{|\xi - Y_0| \leq CR} + \int_{CR \leq |\xi - Y_0| \leq 1} + \int_{|\xi - Y_0| \geq 1}.
$$

Let $\eta := Y - Y_0$. Then $|X - Y| = |(X - Y_0) - \eta|$. Using the inequality

$$
\int_{|\xi - Y_0| \leq CR} \frac{1}{(1 + |\xi|^2)^{\frac{3}{2}}} \frac{C}{R|(\xi - Y_0) - \eta|^2}\, d^2\xi \leq \frac{C}{R} \int_{|\xi| \leq CR} \frac{1}{|\xi|^2 + |\eta|^2}\, d^2\xi
$$

$$
\leq \frac{C}{R} \left| \log \frac{CR}{\text{dist}(Y, \frac{\partial\Omega}{R})} \right|,
$$

we obtain

$$
|\Psi(\widetilde{X}, Y)| \leq \frac{C}{R} \left| \log \frac{CR}{\text{dist}(Y, \frac{\partial\Omega}{R})} \right| + \frac{C}{R} + C,
$$

and

$$
\int_{\substack{\text{dist}(Y, \frac{\partial\Omega}{R}) \leq 1 \\ |Y| \leq 4}} |\Psi(\widetilde{X}, Y)|\, dY \leq \frac{C}{R}(1 + |\log R|).
$$

**Case 2.** For $|Y| \geq 4$, we fix a point $Y = Y_0$. Rescale the variables by $\eta = \frac{Y}{|Y_0|}$ and $\xi = \frac{X}{|Y_0|}$ so that $\xi, \eta \in \frac{\overline{\Omega}}{|Y_0|R}$. Define $g(\xi, \eta) := G(|Y_0|\xi, |Y_0|\eta) = G(X, Y)$. Since $\Delta_X G(X, Y) = \delta(X - Y)$, we have

$$
\Delta_\xi g(\xi, \eta) = \frac{1}{|Y_0|} \delta(\xi - \eta),
$$

by a change of variables. Let $\varphi(\xi, \eta) := |Y_0| \nabla_\eta g(\xi, \eta)$. Then we have

$$
\Delta_\xi \varphi(\xi, \eta) = \nabla_\eta \delta(\xi - \eta) \quad \text{for all } \xi, \eta \in \frac{\Omega}{|Y_0|R},
$$

$$
\varphi(\xi, \eta) = 0 \quad \text{if } \xi \in \frac{\partial\Omega}{|Y_0|R}.
$$

Now, we again divide into two cases.

- If $\text{dist}(\eta, \frac{\partial\Omega}{|Y_0|R}) \geq \frac{1}{10}$, we define $\psi(\xi, \eta)$ satisfying

$$
\varphi(\xi, \eta) = -\frac{1}{4\pi} \nabla_\eta \frac{1}{|\xi - \eta|} + \psi(\xi, \eta).
$$

Since $\Delta_\xi \varphi = \nabla_\eta \delta(\xi - \eta)$, we have

$$\Delta_\xi \psi(\xi, \eta) = 0 \quad \text{for all } \xi, \eta \in \frac{\Omega}{|Y_0|R},$$

$$\psi(\xi, \eta) = \frac{1}{4\pi} \nabla_\eta \frac{1}{|\xi - \eta|} \quad \text{if } \xi \in \frac{\partial\Omega}{|Y_0|R}.$$

By assumption, $|\psi(\xi, \eta)| \leq C$ for all $\xi \in \frac{\partial\Omega}{|Y_0|R}$. By the maximum principle, we have

$$|\psi(\xi, \eta)| \leq C \quad \text{for all } \xi \in \frac{\Omega}{|Y_0|R}.$$

If we apply the boundary regularity theory to the restricted region $\{\xi \in \frac{\Omega}{|Y_0|R}; |\xi| \leq \frac{1}{2}\}$ with $|\eta| = 1$ fixed, then we have

$$|\nabla_\xi \psi(\xi, \eta)| \leq C \quad \text{for all } \xi \in \frac{\Omega}{|Y_0|R}.$$

Hence, we obtain

$$|\varphi(\widetilde{\xi}, \eta)| \leq C \text{dist}(\widetilde{\xi}, \frac{\partial\Omega}{|Y_0|R}) = \frac{C}{|Y_0|},$$

and

$$|\nabla_Y G(\widetilde{X}, Y_0)| \leq \frac{C}{|Y_0|^3}.$$

- If $\text{dist}(\eta, \frac{\partial\Omega}{|Y_0|R}) \leq \frac{1}{10}$, we fix $\eta$, and say $\eta_0$. Let

$$\xi^\star = \xi + 2\text{dist}(\xi, \{\eta \in \mathbb{R}^3; (\eta - \overline{\eta}_0) \cdot \nu(\overline{\eta}_0) = 0\})\nu(\overline{\eta}_0),$$

where $\overline{\eta}_0$ is the boundary point closest to $\eta_0$, and $\nu(\overline{\eta}_0)$ is the outward normal vector at $\overline{\eta}_0$. This means that $\xi^\star$ is the reflection of $\xi$ with respect to the tangent plane at $\overline{\eta}_0$. We define $w(\xi, \eta)$ such that it satisfies

$$\nabla_\xi g(\xi, \eta) = -\frac{1}{4\pi|Y_0|} \nabla_\eta \left( \frac{1}{|\xi - \eta|} - \frac{1}{|\xi^\star - \eta|} \right) + w(\xi, \eta).$$

Then we have

$$\Delta_\xi w(\xi, \eta) = 0 \quad \text{for all } \xi, \eta \in \frac{\Omega}{|Y_0|R},$$

$$w(\xi, \eta) = \frac{1}{4\pi|Y_0|} \left( \frac{\xi - \eta}{|\xi - \eta|^3} - \frac{\xi^\star - \eta}{|\xi^\star - \eta|^3} \right) \quad \text{for } \xi \in \frac{\partial\Omega}{|Y_0|R}.$$

Now, for $\xi \in \frac{\partial\Omega}{|Y_0|R}$, we have

$$|w(\xi, \eta_0)| \leq \frac{C}{|Y_0|} \quad \text{if } |\xi - \overline{\eta}_0| \geq \frac{1}{8},$$

by the triangle inequality, and we have

$$|w(\xi, \eta_0)| \leq \frac{CR|\xi - \overline{\eta}_0|^2}{|\xi - \eta_0|^3}$$

$$\leq \frac{CR|\xi - \overline{\eta}_0|}{|\xi - \overline{\eta}_0|^3 + |\eta_0 - \overline{\eta}_0|^3} \quad \text{if } |\xi - \overline{\eta}_0| \leq \frac{1}{8},$$

by Taylor Theorem. Let $D := \text{dist}(\eta_0, \frac{\partial\Omega}{|Y_0|R})$. We again use the Poisson kernel estimation to get, for $\xi \in B_{\frac{1}{|Y_0|}}(0) \cap \frac{\Omega}{|Y_0|R}$,

$$
|w(\xi, \eta_0)| \leq \int_{|z'| \geq \frac{1}{8}} \frac{\xi_1 - \overline{\eta}_{0,1}}{\{(\xi_1 - \overline{\eta}_{0,1})^2 + |(\xi' - \overline{\eta}_0') - z'|^2\}^{\frac{3}{2}}} \frac{C}{|Y_0|} \, d^2 z'
$$
$$
+ \int_{|z'| \leq \frac{1}{8}} \frac{\xi_1 - \overline{\eta}_{0,1}}{\{(\xi_1 - \overline{\eta}_{0,1})^2 + |(\xi' - \overline{\eta}_0') - z'|^2\}^{\frac{3}{2}}} \frac{CR|z'|}{(|z'|^3 + D^3)} \, d^2 z'
$$
$$
\leq \frac{C}{|Y_0|} + \int_{|Z'| \leq \frac{1}{8D}} \frac{CR|Z'|}{|Z'|^3 + 1} \, d^2 Z'
$$
$$
\leq C\left(\frac{1}{|Y_0|} + R|\log \text{dist}\,(\eta_0, \tfrac{\partial\Omega}{|Y_0|R})|\right).
$$

Here, we use the prime notation to indicate the second and third components in the Cartesian coordinate system. Similarly to the former case, we apply the boundary regularity theory and obtain

$$
|\nabla_\xi w(\xi, \eta_0)| \leq C\left(\frac{1}{|Y_0|} + R|\log \text{dist}\,(\eta_0, \tfrac{\partial\Omega}{|Y_0|R})|\right),
$$

for all $\xi \in B_{\frac{1}{|Y_0|}}(0) \cap \frac{\Omega}{|Y_0|R}$. Hence, by the change of variables, we get

$$
|\nabla_Y G(\widetilde{X}, Y_0)| \leq C\left(\frac{1}{|Y_0|^2} + \frac{R}{|Y_0|^2}|\log \text{dist}\,(\tfrac{Y_0}{|Y_0|}, \tfrac{\partial\Omega}{|Y_0|R})|\right).
$$

In conclusion, if $|Y| \geq 4$, we have

$$
\int_{4 \leq |Y| \leq \frac{L}{R}} |\nabla_Y G(\widetilde{X}, Y)| \, d^3 Y
$$
$$
\leq C \int_{4 \leq |Y| \leq \frac{L}{R}} \left(\frac{1}{|Y|^2} + \frac{R}{|Y|^2}|\log \text{dist}\,(\tfrac{Y}{|Y|}, \tfrac{\partial\Omega}{|Y|R})|\right) d^3 Y.
$$

For the second term, we get

$$
\int_{4 \leq |Y| \leq \frac{L}{R}} \frac{R}{|Y|^2}|\log \text{dist}\,(\tfrac{Y}{|Y|}, \tfrac{\partial\Omega}{|Y_0|R})| \, d^3 Y
$$
$$
\leq C \sum_{n=0}^{\frac{|\log R|}{\log 2}} \int_{4\cdot 2^n \leq |Y| \leq 4\cdot 2^{n+1}} \frac{R}{|Y|^2}|\log \text{dist}\,(\tfrac{Y}{|Y|}, \tfrac{\partial\Omega}{|Y_0|R})| \, d^3 Y
$$
$$
\leq CR \sum_{n=0}^{\frac{|\log R|}{\log 2}} 2^n \int_{4 \leq |Z| \leq 8} |\log \text{dist}\,(\tfrac{Z}{|Z|}, \tfrac{\partial\Omega}{2^n R|Z|})| \, d^3 Z.
$$
$$
\leq C
$$

Notice that the last integration is bounded. Thus, we have established (18) for $|Y| \geq 4$. $\square$

**Lemma 3.2.** *With the same assumptions of Lemma 3.1, we have*

$$\left|\frac{\partial \phi}{\partial x_2}(t, \widetilde{x})\right| + \left|\frac{\partial \phi}{\partial x_3}(t, \widetilde{x})\right| \leq C\widetilde{x}_1(1 + |\log \widetilde{x}_1|),$$

*where $C > 0$ depends only on $L$ and $\|\rho\|_\infty$.*

*Proof.* As in the proof of Lemma 3.1, we take the scaled variables $X = \frac{x}{R}$ and $Y = \frac{y}{R}$ where $R = |\widetilde{x}|$. Say $\widetilde{X} = \frac{\widetilde{x}}{R} = (1, 0, 0)$. To compute $|\frac{\partial G}{\partial x_2}|$, we divide it into two cases.

**Case 1.** If $|Y| \geq 2$, we decompose the Green function $G(X, Y) = \overline{G}(X, Y) + W(X, Y)$ where

$$\overline{G}(X, Y) = -\frac{1}{4\pi R}\left(\frac{1}{|X - Y|} - \frac{1}{|X - Y^*|}\right)$$

and $Y^*$ represents the reflection of $Y$ with respect to the plane $\{X_1 = 0\}$. Notice that if $|X| \leq \frac{3}{4}|Y|$, then we have

$$|\overline{G}(X, Y)| \leq \frac{C}{R|Y|}.$$

Moreover, since $0 \leq W(X, Y) \leq -\overline{G}(X, Y)$, we get

$$|G(X, Y)| \leq \frac{C}{R|Y|}.$$

Now, we take the variables $\xi = \frac{X}{|Y|}$ and $\eta = \frac{Y}{|Y|}$ and consider the restricted region $\Omega_0 = \{\xi \in \frac{\Omega}{R|Y|}; |\xi| \leq \frac{3}{4}\}$ with $Y$ fixed. Since

$$\Delta_\xi G(\xi, \eta) = 0 \text{ in } \Omega_0, \quad |G(\xi, \eta)| \leq \frac{C}{R|Y|} \text{ on } \partial\Omega_0,$$

applying regularity theory leads to

$$\left|\frac{\partial^\alpha G}{\partial \xi^\alpha}(\xi, \eta)\right| \leq \frac{C}{R|Y|} \quad \text{for any multi-index } \alpha.$$

Let $\widetilde{\xi} = \frac{\widetilde{x}}{|Y|}$. Since $|\frac{\partial G}{\partial \xi_2}(0, \eta)| = 0$, we have

$$\left|\frac{\partial G}{\partial \xi_2}(\widetilde{\xi}, \eta)\right| \leq \frac{C}{R|Y|^2},$$

and

$$\left|\frac{\partial G}{\partial X_2}(\widetilde{X}, Y)\right| \leq \frac{C}{R|Y|^3}.$$

**Case 2.** If $|Y| \leq 2$, we denote as $Y_0$ the boundary point closest to $Y$. Decompose $G(X, Y) = \overline{G}(X, Y) + W(X, Y)$ where

$$\overline{G}(X, Y) = -\frac{1}{4\pi R}\left(\frac{1}{|X - Y|} - \frac{1}{|X - \overline{Y}|}\right).$$

Here, $\overline{Y}$ is the reflection point of $Y$ with respect to the tangent plane at $Y_0$. For $X \in \frac{\partial \Omega}{R}$, we express $W(X, Y)$ in terms of $e_\perp := \frac{Y - Y_0}{|Y - Y_0|}$, $\eta := \frac{X - Y_0}{|Y - Y_0|}$, and $\eta^* := \frac{\overline{X} - Y_0}{|Y - Y_0|}$ as

$$W(X, Y) = \frac{1}{4\pi R |Y - Y_0|}\left(\frac{1}{|e_\perp - \eta|} - \frac{1}{|e_\perp - \eta^*|}\right).$$

Then we have

$$\left|\frac{1}{|e_\perp - \eta|} - \frac{1}{|e_\perp - \eta^*|}\right| \begin{cases} \leq CR|Y - Y_0||\eta|^2 & \text{if } |\eta| \leq 1, \\[2ex] \leq \dfrac{CR|Y - Y_0|}{|\eta|} & \text{if } |\eta| \geq 1. \end{cases}$$

Combining these, we obtain $|W(X, Y)| \leq C$. On the other hands, for $X \in \frac{\Omega}{R}$ on the line segment $|X| = 4$, we have $|W(X, Y)| \leq \frac{C}{R}$. Indeed, this can be done by taking $-\overline{G}$ as a supersolution, and then applying the maximum principle. Now, if we consider the region $\frac{\Omega}{R} \cap \{|X| \leq 4\}$ with $Y$ fixed, then using the regularity theory leads to

$$|\nabla_X W(X, Y)| \leq \frac{C}{R},$$

and by adding $|\nabla_X \overline{G}|$ term, we have

$$\left|\frac{\partial G}{\partial X_2}(\widetilde{X}, Y)\right| \leq \frac{C}{R|\widetilde{X} - Y|^2}.$$

From all these calculations, we conclude

$$\int_{\frac{\Omega}{R}} \left|\frac{\partial G}{\partial X_2}(\widetilde{X}, Y)\right| dY = \int_{|Y| \geq 2} + \int_{|Y| \leq 2} \left|\frac{\partial G}{\partial X_2}(\widetilde{X}, Y)\right| dY$$

$$\leq \int_{|Y| \geq 2} \frac{C}{R|Y|^3} \, dY + \int_{|Y| \leq 2} \frac{C}{R|\widetilde{X} - Y|^2} \, dY$$

$$\leq \frac{C}{R}(1 + |\log R|).$$

$\square$

We now give the main result, which plays the role of Velocity Lemma in our setting.

**Lemma 3.3** (Velocity Lemma). *Let $(X(s; t, x, v), V(s; t, x, v))$ be the characteristic curves associated to the Vlasov-Poisson system defined previously. Suppose $\phi(t, x)$ satisfies the assumptions of Lemma 3.1. Then there exist constants $C_1$ and $C_2 > 0$ depending only on $\Omega$, $\|\rho\|_{L^\infty}$, and $\|j\|_{L^\infty}$, such that if $X_\perp$ is small enough, we have*

$$C_1(X_\perp + V_\perp^2)(t) \leq (X_\perp + V_\perp^2)(s) \leq C_2(X_\perp + V_\perp^2)(t),$$

*for $s, t \in [0, T]$.*

*Proof.* Due to Hopf Lemma, we can choose the constant $\epsilon_0 > 0$ such that

(19)
$$\phi(t, x) \leq -\epsilon_0 x_\perp$$

for $x_\perp$ small. We define

$$\alpha(t, x, v) = \frac{v_\perp^2}{2} - \phi(t, x) - \left( \sum_{i=1}^2 \frac{w_i^2 b_i}{1 + k_i x_\perp} \right) x_\perp,$$

where $b_i$'s are the coefficients of the second fundamental form, and $k_i$'s are the principal curvatures of the surface $\partial\Omega$. Notice that $b_i \leq 0$ by the convexity of $\Omega$. So, $\alpha(t, x, v)$ is equivalent to $x_\perp + v_\perp^2$. That is, it is sufficient to show

$$C_1 \alpha(t, X(t), V(t)) \leq \alpha(s, X(s), V(s)) \leq C_2 \alpha(t, X(t), V(t))$$

for $s, t \in [0, T]$. By differentiating $\alpha$ with respect to $t$ along the characteristics and by representing the field $E$ as $E = \nabla_x \phi = E_1 u_1 + E_2 u_2 - E_\perp n_x(\mu_1, \mu_2)$, we have

$$\frac{d\alpha}{dt}(t, x(t), v(t)) = -\frac{\partial\phi}{\partial t}(t, x) - \sum_i w_i E_i \frac{b_i}{k_i}$$

$$- \sum_i \left( 2 w_i b_i E_i + w_i^2 \frac{db_i}{dt} - 2 \sum_{j,l} \frac{\Gamma_{jl}^i w_i w_j w_l b_i}{1 + k_j x_\perp} \right) \frac{x_\perp}{1 + k_i x_\perp}$$

$$+ \sum_i \left( v_\perp k_i - x_\perp \frac{dk_i}{dt} \right) \frac{x_\perp w_i^2 b_i}{(1 + k_i x_\perp)^2},$$

where $\Gamma_{jl}^i$;s are the Christoffel symbols. Using Lemma 3.1, 3.2, and the equation (19), we obtain

$$\left| \frac{d\alpha}{dt}(t, X(t), V(t)) \right| \leq C\alpha(1 + |\log\alpha|).$$

Therefore, the Lemma follows by Gronwall inequality.                □


We give the theorem of well-posedness for the linear problem (1), (3), (4) in the following theorem.

**Theorem 3.4.** *Assume that $E \in C_{t;\, x}^{0;1,\mu}\left([0, T] \times \bar\Omega\right)$ for some $\mu \in (0, 1)$. Suppose that $f_0 \in C_0^{1,\mu}\left(\bar\Omega \times \mathbb{R}^3\right)$ for some $\mu > 0$ and $f_0 \geq 0$. Then there exists a unique solution, $f \in C_{t;(x,v)}^{1;1,\lambda}\left([0, T] \times \Omega \times \mathbb{R}^3\right)$, to the linear Vlasov-Poisson system (1), (3), (4), for some $0 < \lambda < \mu$. Moreover the function $f$ satisfies*

(20)
$$f \geq 0$$

(21)
$$\int f(t, x, v)\, dxdv = \int f_0(x, v)\, dxdv, \quad t \in [0, T].$$

*Proof.* The key point in the proof is that the characteristics (13)-(16) intersect the boundary $\partial\Omega \times \mathbb{R}^3$ at most a finite number of times and they never intersect with the singular set, which is due to Velocity lemma and Lemma 3.3. The essential procedure is similar to the proof of Theorem 2 in [12], we skip the details of the proof. $\qquad\square$

## 4. Iterative approach for the nonlinear problem

In this section, we will show the global existence of classical solutions to the fully nonlinear Vlasov-Poisson system (1)-(5). Since the procedures are similar to ones in [12], we will not try to give every detail of the proofs and instead we refer to [12] whenever we need.

4.1. **Iterative procedure.** We will obtain a solution of the nonlinear system (1)-(5) as the limit of a sequence of functions $f^n$ that are defined by an iterative procedure. More precisely, we define

$$(22) \qquad f^0(t, x, v) = f_0(x, v) \quad , \quad t \geq 0, \ x \in \Omega, \ v \in \mathbb{R}^3$$

$$(23)$$

$$f_t^n + v \cdot \nabla_x f^n + \nabla_x \phi^{n-1} \cdot \nabla_v f^n = 0 \quad , \quad x \in \Omega \subset \mathbb{R}^3 \quad , \ v \in \mathbb{R}^3 \quad , \ t > 0$$

$$(24) \qquad \Delta \phi^{n-1} = \rho^{n-1}(x) \equiv \int_{\mathbb{R}^3} f^{n-1} dv \ , \ x \in \Omega \ , \ t > 0$$

$$(25) \qquad \phi^{n-1} = 0 \ , \ x \in \partial\Omega \ , \ t > 0$$

$$(26) \qquad f^n(0, x, v) = f_0(x, v) \quad x \in \Omega \ , \ v \in \mathbb{R}^3$$

$$(27) \qquad f^n(t, x, v) = f^n(t, x, v^*) \ \ x \in \partial\Omega \ , \ v \in \mathbb{R}^3 \ , \ t > 0$$

for $n = 1, 2, \dots$. We assume that $f_0$ satisfies the nonnegativity condition as well as (10)-(12).

We will use the notation

$$(28) \qquad\qquad\qquad E^n = \nabla\phi^n.$$

The goal is to show that the sequence $f^n$ converges as $n \to \infty$ for all $0 \leq t < \infty$. To this end we need to show as a first step that this sequence is globally defined in time for each $n \geq 0$.

4.2. **The iterative sequence $\{f^n\}$ is globally defined in time.** Given a function $g : \Omega \to \mathbb{R}$, we will denote as $[\cdot]_{0,\lambda;x}$ the seminorm

$$[g]_{0,\lambda;x} \equiv \sup_{x,y\in\Omega} \frac{|g(x) - g(y)|}{|x - y|^{\lambda}}.$$

We define

$$(29) \qquad Q(t) \equiv \sup\{|v| \ | \ (x, v) \in \text{supp } f(s), \ \ 0 \leq s \leq t\}.$$

**Proposition 4.1.** *Let $\mu, \lambda \in (0,1)$, satisfying $\mu > \lambda$. Let $f_0 \in C_0^{1,\mu}\left(\bar{\Omega} \times \mathbb{R}^3\right)$, $f_0 \geq 0$ satisfy (12). Then, the sequence of functions $f^n$ is globally defined for each $x \in \Omega$, $v \in \mathbb{R}^3$ and $0 \leq t < \infty$. Moreover we have $f^n \in C_{t;(x,v)}^{1;1,\lambda}\left([0,T] \times \Omega \times \mathbb{R}^3\right)$ for any $T > 0$ and $\|f^n\|_\infty = \|f_0\|_\infty$, $\int \rho_n(x,t)\, dx = \int f_0(t,x,v)\, dxdv$.*

*Proof.* This proposition can be proved using Theorem 3.4 and by induction on $n$. We omit the details. $\square$

### 4.3. The sequence $\{f^n\}$ converges to a solution of the VP system if the sequence $\{Q^n\}$ is bounded. We define the following measure for the maximal velocities reached for the distribution $f^n$

$$(30) \qquad Q^n(t) \equiv \sup\left\{|v| \ \mid \ (x,v) \in \operatorname{supp} f^n(s), \ \ 0 \leq s \leq t\right\}.$$

**Proposition 4.2.** *Under the assumptions of Theorem 2.2, suppose that $Q^n(t) \leq K$ for $n \geq n_0$, $0 \leq t \leq T$. Then, $f^n \to f$ in $C_{t;(x,v)}^{\nu;1,\lambda}\left([0,T] \times \Omega \times \mathbb{R}^3\right)$ as $n \to \infty$ with $0 < \lambda < \mu$, $0 < \nu < 1$ and where $f \in C_{t;(x,v)}^{1;1,\lambda}\left([0,T] \times \Omega \times \mathbb{R}^3\right)$ is a solution of (1)-(5).*

*Proof.* The proof of the Proposition 4.2 is similar to that of Proposition 3 in [12] and we omit it. $\square$

### 4.4. Prolongability of uniform estimates for the functions $f^n$.

**Proposition 4.3.** *Let $Q^n$, $Q$ be as in (30), (29) respectively. Suppose that $\max\left\{\sup_{n \geq n_0} Q^n(t), Q(t)\right\} \leq K$ for $0 \leq t \leq T$. We also assume that $f^n \to f$ in $C_{t;(x,v)}^{\nu;1,\lambda}\left([0,T] \times \Omega \times \mathbb{R}^3\right)$ for any $0 < \lambda < \mu$, $0 < \nu < 1$. Then $\lim_{n\to\infty} Q^n(t) = Q(t)$ uniformly on $[0,T]$.*

*Proof.* The proof relies on Velocity lemma, Lemma 3.3. The characteristics starting in $\alpha(0) \geq C\delta_0$ remain during their evolution in the set $\{\alpha(t) \geq C\delta_0\}$ due to Lemma 3.3. Therefore, these characteristics remain separated from the singular set from which we can deduce the convergence of $Q^n(t)$ to $Q(t)$ as $n \to \infty$. For the details, refer to [12]. $\square$

The following Proposition concerns the prolongability of the uniform estimates on $Q^n(t)$ and we skip the proof.

**Proposition 4.4.** *Suppose that for some $T \geq 0$ there exist $K > 0$ and $n_0 \geq 0$ such that for any $n \geq n_0$ and $0 \leq t \leq T$ we have $Q^n(t) \leq K$. Then, there exists $\varepsilon_0 = \varepsilon_0(K, \|f_0\|_\infty) > 0$ such that for $0 \leq t \leq T + \varepsilon_0$ and $n \geq n_0$ the following estimate holds*

$$Q^n(t) \leq 2K.$$

We give in the following some of basic energy estimates for the Vlasov-Poisson system (cf. [4]). Proofs are standard in kinetic theory and we omit them.

**Proposition 4.5.** *Suppose that $f$ is a solution of (1)-(5) defined in $0 \leq t \leq T$ with $f(0, x, v) = f_0(x, v)$, where $f_0 \geq 0$. There exists $C$ depending only on $T$ and on the regularity norms assumed for $f_0$ in Theorem 2.2 such that*

$$(31) \qquad \sup_{0 \leq t \leq T} \int_\Omega v^2 f(x, t) \, dv dx \leq C$$

$$(32) \qquad \sup_{0 \leq t \leq T} \left[ \|\rho(t, \cdot)\|_{L^{\frac{5}{3}}(\Omega)} \right] \leq C$$

$$(33) \qquad \|f(t)\|_{L^p(\Omega \times \mathbb{R}^3)} = \|f_0\|_{L^p(\Omega \times \mathbb{R}^3)} \ , \quad \text{for all } 1 \leq p \leq \infty.$$

$$(34) \qquad \frac{d}{dt} \left( \int_{\Omega \times \mathbb{R}^3} v^2 f \, dx dv + \int_\Omega (E)^2 \, dx \right) = 0$$

## 5. GLOBAL BOUND FOR $Q(t)$.

In this section we show that the function $Q(t)$ can be bounded in any time interval $0 \leq t \leq T$ and therefore that the corresponding solutions of (1)-(5) can be extended to arbitrarily long intervals. The global-in-time bound on $Q(t)$ was first proved by Pfaffelmoser (cf. [16]) in the case of the whole space and the method of Pfaffelmoser has been adapted to the case of bounded domains with purely reflected boundary conditions at $\partial\Omega$ (cf. [12]). The main content of the result is a uniform estimate for $Q(t)$ as long as $f$ is defined.

From the definition (29) of $Q(t)$, we obtain the following estimate

$$(35) \qquad \|\rho\|_\infty \leq \|f\|_\infty Q(t)^3.$$

where $\rho$ is in (2).

The main result of this section is given in the following. Since the theorem and its proof do not depend on the boundary conditions for the electric potential $\phi$ (whether Dirichlet or Neumann), they can be proved as in Theorem 3 in [12]. We state the theorem without its proof and without auxiliary lemmas.

**Theorem 5.1.** *Let* $f_0 \in C^{1,\mu} \left( \Omega \times \mathbb{R}^3 \right)$ *with* $0 < \mu < 1$. *Suppose that* $f \in C^{1;1,\lambda}_{t,(x,v)} \left( [0,T] \times \Omega \times \mathbb{R}^3 \right)$ *is a solution of (1)-(5) with* $\lambda \in (0,1)$, $0 < T < \infty$. *There exists* $\sigma(T) < \infty$ *depending only on* $T$, $Q(0)$, *and* $\|f_0\|_{C^{1,\mu}(\Omega \times \mathbb{R}^3)}$ *such that*

$$(36) \qquad\qquad Q(t) \leq \sigma(T) \quad, \quad 0 \leq t \leq T.$$

***Proof of Theorem 2.2.*** The proof is similar to that in [12] and we omit it. $\qquad\square$

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